immediately obvious that there need be no relationship between boundary and domain weighting functions. Furthermore, as shown in Ref. 2, there need be no relationship between weighting functions and the trial functions which make up the approximate solution. These observations can be demonstrated with the following simple example of a particle moving with constant acceleration.

$$\text{for } t>0: \quad \ddot{y}=c \tag{1}$$

$$\text{for } t=0: \quad y=y_0, \quad \dot{y}=v_0 \tag{2}$$

A weighted residuals formulation begins with the equation

$$\int_0^t (\ddot{y}-c)\psi \mathrm{d}t + [y(0)-y_0]\alpha + [\dot{y}(0)-v_0]\beta = 0 \tag{3}$$

Assume

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \tag{4}$$

$$\psi(t) = b_1 + b_2 \sin t + b_3 \cos t \tag{5}$$

and require that Eq. (3) be satisfied for arbitrary values of b_1, b_2, b_3, α, and β. The results are

$$a_0 = y_0, \quad a_1 = v_0, \quad a_2 = \tfrac{1}{2}c, \quad a_3 = 0, \quad a_4 = 0$$

which gives

$$y(t) = y_0 + v_0 t + \tfrac{1}{2}ct^2 \tag{6}$$

Equation (6) is the correct answer, valid for all times t, as determined from a weighted residuals formulation in which the boundary weighting functions α and β are independent of the domain weighting function ψ and the ψ function does not have the same form as the trial function.

Simkins' Eq. (4), with obvious changes in notation, can also be used to solve the simple problem of Eqs. (1) and (2) above, with the trial solution of Eq. (4) above. It makes absolutely no difference in the final result whether one sets

$$\delta\lambda_1 = \pm\delta y'(0) \text{ and } \delta\lambda_2 = \pm\delta y(0) \text{ or } \delta\lambda_1 = \pm\delta y(0)$$
$$\text{and } \delta\lambda_2 = \pm\delta y'(0)$$

Each of the eight possibilities gives the exact results of Eq. (6) above.

This Comment is not suggesting that there is normally any advantage associated with such generality in weighting functions. The point is that the choices of Ref. 1 are not unique. Of course there is nothing wrong with associating boundary and domain weighting functions. Sometimes this makes it possible to achieve a formulation which reduces to a true variational formulation which might exist for a similar problem, and this might be desirable. Perhaps the search for the most accurate approximation according to some measure might provide guidance for choosing the Lagrange multipliers. However, as the example problem above shows, there is nothing really new or profound in the so-called unconstrained variational statements of Ref. 1. What we have are some very nice examples of the straightforward application of the method of weighted residuals applied to problems in a space-time domain.

Finally, a comment concerning the author's Eq. (12), which is stated to be valid for an elastic solid. More precisely, the indicated equality is valid only for a material with linear stress-strain laws experiencing sufficiently small deformations so that linear strain-displacement relations are satisfactory.

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Reply by Author to C. V. Smith Jr.

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MANY references can be cited to demonstrate that the existence of a functional is not required to state a physical problem in variational form. One such example is Hamilton's Law of Varying Action[1]:

$$\delta\int_{t_0}^{t_1} (T+W)\mathrm{d}t - \frac{\partial T}{\partial \dot{q}_i}\delta q_i \Big]_{t_0}^{t_1} = 0$$

which only becomes the extremum principle of Hamilton if the second term vanishes. Even the principle of virtual work[2] is generally not an extremum principle. Still other examples may be found in the text of Leipholz.[3]

Lanczos (Ref. 2, p. 66), however, gives the clearest contradiction to Smith's understanding that only $\delta I=0$ represents a meaningful variational statement: "A variational problem with non-holonomic auxiliary conditions cannot be reduced to a form where the variation of a certain quantity is put equal to zero. However, the equations of motion are once more derivable, with the help of the [Lagrange] multiplier method, in a fashion analogous to the case of holonomic conditions."

Lanczos offers no explanation as to how the Lagrange multipliers are to be determined for general nonholonmic systems. The application of the Lagrange multiplier method to the nonholonomic initial conditions was, in fact, the subject of my paper.[4] The result presented is a variational formulation applicable to a large class of initial boundary-value problems. Upon performing the indicated variations, the appropriate equations of motion together with all boundary and initial conditions, are generated—indicating that the formulation represents a correct physical balance and may be useful as a basis for obtaining approximate solutions to such problems.

In reply to Smith's contention that it is really the method of weighted residuals (MWR) which has been utilized, one should realize that MWR requires that a choice be made as to how the residuals are to be weighted and what weighting functions are to be employed. Although there is no proof that best or even unique results will be obtained, many prefer that these choices be dictated by physical considerations. A variational statement representing a physical law or balance makes these choices *automatically*. This in itself is sufficient justification for distinguishing between variational methods (with or without a functional) and MWR. It is not surprising that different weighting functions, whether they have been introduced arbitrarily or by variational methods, can be used to achieve approximate and occasionally even exact solutions. Smith's example, however, is not a good one, since his assumed polynomial approximation has the form of the exact solution to begin with.

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The use of δ-notation in constructing variational formulations from partial differential equations is by now standard.[5] In addition to its general utility as an operator, the δ symbol has physical connotations, e.g., as in virtual displacements or virtual work, the theory of which preceded that of the variational calculus.

Finally, Smith's observation that Eq. (12) applies only in case of linear elasticity is well taken. The omission was unfortunate but obvious. It should be mentioned, however, that Tiersten,[6] who first derived the most general form of Eq. (14), apparently did not require this restriction. Thus the variational formulation [Eq. (14)] should be considered applicable to any Lagrange density of the form:

$$L = \tfrac{1}{2}\rho \dot{u}_k \dot{u}_k - U(u_k, u_{k,l}, x_j)$$

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[3]Leipholz, H., *Direct Variational Methods and Eigenvalue Problems in Engineering*, Noordhoff International Publishers, 1977, p. 55.

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